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Wavelet Simultaneous Approximation from Samples Affected by Noise

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Abstract—The authors recently introduced a p^{th} order wavelet regularization method together with the GCV criterion for approximating a function from a finite sample affected by noise. Convergence results of the method were proven for the L_2-norm. The present paper addresses the problem of simultaneous approximation, which is of interest in applications, where derivatives are useful for extracting several features from a signal. It is proved that it is not needed to devise a special method to this purpose, since the convergence of the method devised by the authors also works for the H_q-norm, $0 \leq q < p$. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Wavelets, Simultaneous approximation, Signal processing.

1. INTRODUCTION

Smoothing data is an important problem in mathematics, due to the large number of practical applications involved, where data, generally coming from measurements, are corrupted by noise. In the mathematical framework, the smoothing data problem can be recast as the problem of approximating a function when a finite sample is affected by noise. While the engineering literature is plenty of methods that face the smoothing data problem, the mathematical literature in this respect is not as rich as for the classical problem of approximating a function when noise is not taken into account. This is due to the difficulty in setting a rigorous mathematical theory able to give convergence proofs in the presence of noise. We recall the results obtained by Wahba (see [1] for an excellent summarization of her results) when raw data (i.e., not transformed) are considered. The recent interest on wavelet theory made that the problem of smoothing data was also reconsidered with the aid of wavelets. Several results are available in the framework of statistical analysis (e.g., [2–4]), approximation theory [5] and regularization [6,7]. In those papers a (not always fully rigorous) mathematical analysis of the proposed methods is given. Nevertheless, effort is made to prove results that have an immediate fallout in applications (e.g., convergence for finite noise, convergence of objective criteria for choosing particular parameters).

An important problem arising from applications is the estimate of the derivatives of a function from inexact data. Instances of this problem concern picking boundaries of objects in images

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX (first derivative) and estimating the curvature of a function (second derivative). The present paper deals just with this problem. To this purpose, we shall consider the wavelet regularization method analyzed in [6] and shall prove that, under suitable conditions, it is able to ensure the convergence to the true derivatives, beside the convergence to the true function. Moreover, it will be shown that the method solves the problem in an optimal way asymptotically.

The paper is organized as follows. In Section 2, we present the method and give key results already obtained by the authors that will be recalled in the rest of the paper. Section 3 deals with convergence to the function and to its derivatives. Section 4 addresses the point of devising a completely objective method (where no exogenous information is introduced in the problem, apart data). Finally, numerical experiments are worked out in Section 5 in order to show the performance of the method.

2. PRELIMINARIES AND NOTATION

Let us consider the smoothing problem

$$y_i = f_i + \varepsilon_i, \qquad 0 \le i < N, \tag{1}$$

where $N = 2^J$, $\underline{f} \equiv \{f_0, \dots, f_{N-1}\} \equiv \{F(x_0), \dots, F(x_{N-1})\}$, $x_i = i/N$, $\underline{\varepsilon} \equiv \{\varepsilon_0, \dots, \varepsilon_{N-1}\}$ is white noise (independently, $N(0, \sigma^2)$-distributed random variables), with variance σ^2 supposed unknown.

The aim of the smoothing problem is to retrieve a (finite dimensional) approximation of F starting from a (noised) sample at the finite set of points x_i, $0 \le i < N$ (we suppose without any loss of generality F periodic on $[0,1]$).

Let us consider the (orthogonal) periodic wavelet expansion of F,

$$F(x) = F_{-1,0}\varphi(x) + \sum_{j \ge 0} \sum_{k=0}^{2^j-1} F_{j,k}\psi_{j,k}(x), \tag{2}$$

where φ is the scaling function, ψ is the wavelet function and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ (for the sake of brevity in the following, we shall set $a_j = \max(2^j, 1)$, so that indices j, k will range in $j \ge -1$ and $0 \le k < a_j$).

For nonperiodic functions a finite interval orthogonal transform can be considered according to [8].

Recently, the authors [6,7] independently introduced a wavelet regularization method for smoothing data based on the minimization of the following functional,

$$\min_{F^{N,\lambda}} \left\| F^{N,\lambda} - Y^N \right\|_2^2 + \lambda \left\| F^{N,\lambda} \right\|_{H^p}^2, \tag{3}$$

where λ is a suitable regularization parameter, H^p is the Sobolev space $\{F | (1 - \frac{d^2}{dx^2})^{p/2} F \in L^2(0,1)\}$, and $F^{N,\lambda}$, Y^N are finite dimensional approximations of the regularized solution and of the noised function, respectively. If functions are expanded in an (orthogonal) wavelet basis as in equation (2) (dropped at $j = J-1$), then the solution of the regularization problem (3) is [6, Theorem 2.2]

$$F_{j,k}^{N,\lambda} = \frac{Y_{j,k}}{1 + \lambda a_j^{2p}}, \qquad -1 \le j < J, \quad 0 \le k < a_j. \tag{4}$$

The choice of the regularization parameter is crucial in most regularization problems. When the variance of the noise affecting the data is unknown, the GCV criterion [1] is known to be one of the most effective methods for estimating the regularization parameter, and indeed it has been used in [6] according to the expression

$$\min_{\lambda\geq 0}\mathrm{GCV}_N(\lambda),\quad \mathrm{GCV}_N(\lambda)=\frac{\sum_{j=-1}^{J-1}\sum_{k=0}^{a_j-1}\left(F_{j,k}^{N,\lambda}-Y_{j,k}\right)^2}{\left[1/N\sum_{j=-1}^{J-1}\sum_{k=0}^{a_j-1}\left(1-1/\left(1+\lambda a_j^{2p}\right)\right)\right]^2}. \tag{5}$$

Summarizing, the procedure developed by the authors goes through the following steps.

STEP 1. Compute the discrete wavelet transform of $2^{-J/2}\underline{y}\equiv\{2^{-J/2}y_{j,k},\ -1\leq j<J,\ 0\leq k<a_j\}$. Due to the linearity of the transform it follows $y_{j,k}=f_{j,k}+\eta_{j,k}$, with $f_{j,k}$ and $\eta_{j,k}$ being the discrete wavelet transform of $\underline{f}$ and $\underline{\varepsilon}$, respectively. We assume

$$f_{j,k}\simeq F_{j,k} \tag{6}$$

(and also $y_{j,k}\simeq Y_{j,k}$). The approximation (6) is used in all applications and justified by several papers [9,10]; it is even better in the case Coiflets are considered as a basis [11]. Due to the orthonormality of the wavelet transform, $\eta_{j,k}$ are also independently distributed as $N(0,\sigma^2/N)$.

STEP 2. Choose the regularization parameter according to the GCV criterion (5).

STEP 3. Regularize the wavelet transform of Step 1 by equation (4), with the regularization parameter given in Step 2, yielding $F_{j,k}^{N,\lambda}$, $-1\leq j<J$, $0\leq k<a_j$.

STEP 4. Compute the discrete inverse wavelet transform of $2^{J/2}F_{j,k}^{N,\lambda}$, and take it as the regularized solution of the smoothing problem (1) at the same nodes.

We define

$$T_N(\lambda)=E\left\|F^{N,\lambda}-Y^N\right\|_2^2,$$

and

$$V_N(\lambda)=E\mathrm{GCV}_N(\lambda),$$

where E means taking the expected value. The following two results have been proved.

THEOREM 3.1. *Let $F\in H^p$, $p>1/4$. Then*

$$E\left[\|F^{N,\lambda}-F\|^2_{L^2(0,1)}\right]\leq\left(\frac{\sigma^2}{N}\right)^{(2p)/(2p+1)}\left[D(p)\|F\|_{H^p}^{2/(2p+1)}+\left(\frac{\sigma^2}{N}\right)^{1/(2p+1)}\right]+\frac{\|F\|^2_{H^p}}{N^{2p}}$$

for $\lambda=C(p)(\sigma^2/(\|F\|^2_{H^p}N))^{(2p)/(2p+1)}$, where $C(p)$ and $D(p)$ do not depend on N.

THEOREM 4.6. *The choice of λ provided by the GCV_N is asymptotically optimal in the average, in the sense that if λ_N is any minimizer of $GCV_N(\lambda)$, then*

$$\lim_{N\to\infty}\frac{T_N(\lambda_N)}{\min_{\lambda\geq 0}T_N(\lambda)}=1.$$

As mentioned in the Introduction, the aim of the paper is to consider the convergence not only of the function but also of (some of) its derivatives. To this purpose the most natural framework is convergence in Sobolev spaces H^p with respect to the norm defined as [12]

$$\|F\|^2_{H^q}=\sum_{j\geq -1}\sum_{k=0}^{a_j-1}a_j^{2q}F_{j,k}^2.$$

In the present paper, we prove that it is not needed to develop another regularization problem and a corresponding GCV criterion to this purpose, as the method developed in [6] is indeed able to deal with this case.

In next sections, we shall consider quantities A, B depending on the variables λ, J, and N and possibly on other parameters. We shall indicate by $A\approx B$ that the inequalities $C_1B\leq A\leq C_2B$ hold with some $0<C_1,C_2<\infty$ not depending on λ, J, and N but possibly depending on the other parameters. The letter C will denote a generic quantity with the same dependence as C_1, C_2 with respect to variables and parameters.

3. CONVERGENCE

We first state the fundamental.

LEMMA 3.1.
$$\sum_{j=-1}^{J-1} \frac{a_j^r}{\left(1+a_j^s\lambda\right)^k} \approx \frac{1}{\lambda^{r/s}} \int_{\lambda^{r/s}}^{a_J^r \lambda^{r/s}} \frac{dx}{\left(1+x^{s/r}\right)^k}.$$

PROOF. Let $f_\lambda(x) = (1+x^{s/r})^{-k}$. We have

$$\begin{aligned}\frac{1}{\lambda^{r/s}} \int_{\lambda^{r/s}}^{a_J^r \lambda^{r/s}} f_1(x)\,dx &= \int_1^{a_J^r} f_1\left(\lambda^{r/s} x\right)\,dx = \int_1^{a_J^r} f_\lambda(x)\,dx \\ &= \sum_{j=-1}^{J-1} \int_{a_j^r}^{a_{j+1}^r} f_\lambda(x)\,dx.\end{aligned}$$

The result follows from

$$\left(a_{j+1}^r - a_j^r\right) f_\lambda\left(a_{j+1}^r\right) \le \int_{a_j^r}^{a_{j+1}^r} f_\lambda(x)\,dx \le \left(a_{j+1}^r - a_j^r\right) f_\lambda\left(a_j^r\right). \qquad \blacksquare$$

Set

$$\begin{aligned}T_{N,q}(\lambda) &= E \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} a_j^{2q} \left(F_{j,k} - \frac{F_{j,k}+\varepsilon_{j,k}}{1+a_j^{2p}\lambda}\right)^2 \\ &= \lambda^2 \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} \frac{a_j^{4p+2q} F_{j,k}^2}{\left(1+a_j^{2p}\lambda\right)^2} + \frac{\sigma^2}{N} \sum_{j=-1}^{J-1} \frac{a_j^{2q+1}}{\left(1+a_j^{2p}\lambda\right)^2}. \qquad \blacksquare\end{aligned}$$

In the following we shall assume $0 \le q < p$.

LEMMA 3.2. *We have*
$$T_{N,q}(\lambda) \le \frac{\lambda}{4} \|F\|_{H_{p+q}}^2 + \frac{C\sigma^2}{N\lambda^{(2q+1)/2p}}.$$
Therefore, if $\lambda_N \ge CN^{-\alpha}$ with $0 < \alpha < 2p/(2q+1)$, then $\lim_{N\to\infty} T_{N,q}(\lambda_N) = 0$. In particular,

$$\lim_{N\to\infty} \min_{\lambda\ge 0} T_{N,q}(\lambda) = 0.$$

PROOF. $(1+a_j^{2p}\lambda)^2 \ge 4a_j^{2p}\lambda$ so that

$$\lambda^2 \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} \frac{a_j^{4p+2q} F_{j,k}^2}{\left(1+a_j^{2p}\lambda\right)^2} \le \frac{\lambda}{4} \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} a_j^{2(p+q)} F_{j,k}^2.$$

From Lemma 3.1

$$\sum_{j=-1}^{J-1} \frac{a_j^{2q+1}}{\left(1+a_j^{2p}\lambda\right)^2} \le \frac{C}{\lambda^{(2q+1)/2p}} \int_0^\infty \frac{dx}{\left(1+x^{2p/(2q+1)}\right)^2}.$$

The integral is finite since
$$\frac{2p}{2q+1} \ge \frac{2p}{2p+1} > \frac{1}{2}. \qquad \blacksquare$$

Let $\lambda_{N,q}$ be the minimizer of $T_{N,q}$.

Lemma 3.3. *If $F \neq 0$, then $\lim_{N\to\infty} \lambda_{N,q} = 0$.*

Proof. Since $F \neq 0$, there are j_0, k_0 so that $F_{j_0,k_0} \neq 0$. Hence,

$$T_{N,q}(\lambda) \geq \frac{\lambda^2 a_{j_0}^{4p+2q} F_{j_0,k_0}^2}{\left(1 + a_{j_0}^{2p}\lambda\right)^2}. \tag{7}$$

By Lemma 3.2, $\lim_{N\to\infty} \min_{\lambda\geq 0} T_{N,q}(\lambda) = 0$. Hence, (7) implies $\lim_{N\to\infty} \lambda_{N,q} = 0$. ∎

Lemma 3.4.

$$\lim_{N\to\infty} N\lambda_{N,q}^{(2q+1)/2p} = \infty.$$

Proof. By Lemma 3.1,

$$T_{N,q}(\lambda) \geq \frac{C}{N\lambda^{(2q+1)/2p}} \int_{\lambda^{(2q+1)/2p}}^{\left(N\lambda^{1/2p}\right)^{2q+1}} f(x)\,dx, \tag{8}$$

where $f(x) = (1 + x^{2p/(2q+1)})^{-2}$. We first show that $N\lambda_{N,q}^{1/2p} \geq C > 0$. If not, we would have $N\lambda_{N,q}^{1/2p} \to 0$ by passing to a subsequence. But then

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N\lambda^{(2q+1)/2p}} \int_{\lambda^{(2q+1)/2p}}^{\left(N\lambda^{1/2p}\right)^{2q+1}} f(x)\,dx \\
&= \lim_{N\to\infty} \frac{N^{2q+1}-1}{N} \frac{1}{\left(N\lambda^{1/2p}\right)^{2q+1} - \lambda^{(2q+1)/2p}} \int_{\lambda^{(2q+1)/2p}}^{\left(N\lambda^{1/2p}\right)^{2q+1}} f(x)\,dx \\
&= f(0) \lim_{N\to\infty} \frac{N^{2q+1}-1}{N} > 0,
\end{aligned}$$

which contradicts $T_{N,q}(\lambda_{N,q}) \to 0$.

Since $\lambda_{N,q} \to 0$, we have that the integral in (8) is bounded from below by $C > 0$, so that $N\lambda_{N,q}^{(2q+1)/2p} \to \infty$. ∎

Write the derivative of $T_{N,q}(\lambda)$ as $T_{N,q}^1(\lambda) - T_{N,q}^2(\lambda)$, where

$$T_{N,q}^1(\lambda) = 2\lambda \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} \frac{a_j^{4p+2q} F_{j,k}^2}{\left(1 + a_j^{2p}\lambda\right)^3}, \qquad T_{N,q}^2(\lambda) = \frac{2\sigma^2}{N} \sum_{j=-1}^{J-1} \frac{a_j^{2p+2q+1}}{\left(1 + a_j^{2p}\lambda\right)^3}.$$

Lemma 3.5.

(a) *If $F \in H_{p+q}$, then*

$$T_{N,q}^1(\lambda) \leq 2\,\|F\|_{H_{p+q}}^2 \quad \text{and} \quad \lim_{\lambda\to 0} T_{N,q}^1(\lambda) = 0;$$

(b) $T_{N,q}^1(\lambda) \leq 2\lambda \|F\|_{H_{2p+q}}^2$;

(c)

$$T_{N,q}^2(\lambda) \leq \frac{C}{N\lambda^{(2p+2q+1)/2p}} \quad \text{and} \quad T_{N,q}^2(\lambda_{N,q}) \approx \frac{1}{N\lambda_{N,q}^{(2p+2q+1)/2p}}.$$

Proof.

(a)

$$T_{N,q}^1(\lambda) \leq 2 \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} \frac{a_j^{2p}\lambda}{1 + a_j^{2p}\lambda} a_j^{2(p+q)} F_{j,k}^2 \leq 2\,\|F\|_{H_{p+q}}^2.$$

By Lebesgue's dominated convergence theorem, $\lim_{\lambda\to 0} T^1_{N,q}(\lambda) = 0$.

(b)

$$T^1_{N,q}(\lambda) \le 2\lambda \sum_{j=-1}^{J-1} \sum_{k=0}^{a_j-1} a_j^{2(2p+q)} F^2_{j,k}.$$

(c) By Lemma 3.1,

$$T^2_{N,q}(\lambda) \le \frac{C}{N\lambda^{(2p+2q+1)/2p}} \int_0^\infty \frac{dx}{\left(1+x^{2p/(2p+2q+1)}\right)^3}.$$

The integral is finite since

$$\frac{2p}{2p+2q+1} \ge \frac{2p}{4p+1} > \frac{1}{3}.$$

Also by Lemma 3.1,

$$T^2_{N,q}(\lambda_{N,q}) \ge \frac{C}{N\lambda_{N,q}^{(2p+2q+1)/2p}} \int_{\lambda_{N,q}^{(2p+2q+1)/2p}}^{\left(N\lambda_{N,q}^{1/2p}\right)^{2p+2q+1}} \frac{dx}{\left(1+x^{2p/(2p+2q+1)}\right)^3}.$$

By Lemmas 3.3 and 3.4, the lower limit in the integral converges to 0, while the upper limit converges to ∞. Hence,

$$T^2_{N,q}(\lambda_{N,q}) \ge \frac{C}{N\lambda_{N,q}^{(2p+2q+1)/2p}}.$$ ∎

The following proposition gives the order of the minimizer $\lambda_{N,q}$.

PROPOSITION 3.1. *If $F \in H_{p+q}\backslash\{0\}$, then*

$$CN^{(-2p)(2p+2q+1)} \le \lambda_{N,q} \le CN^{(-2p)/(4p+2q+1)}.$$

If $F \in H_{2p+q}\backslash\{0\}$, then

$$\lambda_{N,q} \approx N^{(-2p)/(4p+2q+1)}.$$

PROOF.

$$\frac{C}{N\lambda_{N,q}^{(2p+2q+1)/2p}} \le T^2_{N,q}(\lambda_{N,q}) = T^1_{N,q}(\lambda_{N,q}) \le 2\,\|F\|^2_{H_{p+q}}, \tag{9}$$

so that

$$\lambda_{N,q} \ge CN^{-2p/(2p+2q+1)}.$$

If $F \in H_{2p+q}$, then $T^1_{N,q}(\lambda_{N,q}) \le C\lambda_{N,q}$, so that by (9)

$$\lambda_{N,q} \ge CN^{-2p/(4p+2q+1)}.$$

As $F \neq 0$ we have $T^1_{N,q}(\lambda_{N,q}) \ge C\lambda_{N,q}$, so that

$$\lambda_{N,q} \le CN^{-2p/(4p+2q+1)}.$$ ∎

From the above result it follows that for the minimizer $\lambda_{N,0}$ of the regularization problem with respect to the H_0 norm (i.e., the usual L_2-norm regularization dealt in [6]), we have the following.

COROLLARY 3.1. *If $F \in H_p\backslash\{0\}$, then*

$$CN^{-2p/(2p+1)} \le \lambda_{N,0} \le CN^{-2p/(4p+1)}. \tag{10}$$

If $F \in H_{2p}\backslash\{0\}$, then

$$\lambda_{N,0} \approx N^{-2p/(4p+1)}. \tag{11}$$

COROLLARY 3.2. *If $F \in H_{2p+q}\backslash\{0\}$, then*

$$\frac{\lambda_{N,q}}{\lambda_{N,0}} \approx N^{2p/(4p+1)-2p/(4p+2q+1)} = N^{4pq/(4p+2q+1)(4p+1)}. \tag{12}$$

Corollary 3.2 is important because it says that $\lambda_{N,q}$ and $\lambda_{N,0}$ asymptotically split apart. However, we shall see that this is not a problem both from the practical point of view (we defer discussion on this topic to the Section 5) and from the theoretical point of view. First, we state the following.

LEMMA 3.6. *If $F \in H_{p+q}\backslash\{0\}$, then*

$$N\lambda_{N,0}^{(2p+2q+1)/2p} \to \infty.$$

PROOF. By Lemma 3.5

$$\frac{C}{N\lambda_{N,0}^{(2p+1)/2p}} \le T^2_{N,0}(\lambda_{N,0}) = T^1_{N,0}(\lambda_{N,0}),$$

hence

$$\frac{C}{N\lambda_{N,0}^{(2p+2q+1)/2p}} \le \frac{1}{\lambda_{N,0}^{q/p}} T^1_{N,0}(\lambda_{N,0}).$$

The proof will be complete if we show that

$$\lim_{\lambda\to 0} \frac{1}{\lambda^{q/p}} T^1_{N,0}(\lambda) = 0.$$

Given $\varepsilon > 0$, we may find J_ε so that $\sum_{k=0}^{a_j-1} a_j^{2(p+q)} F^2_{j,k} \le \varepsilon$ for $j \ge J_\varepsilon$. We have

$$T^1_{N,0}(\lambda) = 2\lambda \sum_{j=0}^{J_\varepsilon-1} \sum_{k=0}^{a_j-1} \frac{a_j^{4p} F^2_{j,k}}{\left(1+a_j^{2p}\lambda\right)^3} + 2\lambda \sum_{j=J_\varepsilon}^{J-1} \sum_{k=0}^{a_j-1} \frac{a_j^{4p} F^2_{j,k}}{\left(1+a_j^{2p}\lambda\right)^3}.$$

When multiplied by $\lambda^{-q/p}$ the first term reads

$$2\lambda^{1-q/p} \sum_{j=0}^{J_\varepsilon-1} \sum_{k=0}^{a_j-1} \frac{a_j^{4p} F^2_{j,k}}{\left(1+a_j^{2p}\lambda\right)^3}$$

which converges to 0 as the sum is finite. For the second term, we have by Lemma 3.1

$$\begin{aligned}\sum_{j=J_\varepsilon}^{J-1} \sum_{k=0}^{a_j-1} \frac{a_j^{2(p-q)} a_j^{2(p+q)} F^2_{j,k}}{\left(1+a_j^{2p}\lambda\right)^3} &\le \varepsilon \sum_{j=-1}^{J-1} \frac{a_j^{2(p-q)}}{\left(1+a_j^{2p}\lambda\right)^3} \\ &\le \frac{C\varepsilon}{\lambda^{1-q/p}} \int_0^\infty \frac{dx}{\left(1+x^{2p/(2(p-q))}\right)}.\end{aligned}$$

The integral is finite as

$$\frac{2p}{2(p-q)} \ge 1.$$

Hence, the second sum multiplied by $\lambda^{-q/p}$ is $\le C\varepsilon$. ∎

Finally, we are able to prove the following.

PROPOSITION 3.2. *If $F \in H_{p+q}\backslash\{0\}$, then*

$$T_{N,q}(\lambda_{N,0}) \to 0 \quad \text{and} \quad T'_{N,q}(\lambda_{N,0}) \to 0.$$

PROOF. By Lemma 3.2

$$T_{N,q}(\lambda_{N,0}) \leq C\lambda_{N,0} + \frac{C}{N\lambda_{N,0}^{(2q+1)/2p}}.$$

By Lemma 3.6,

$$N\lambda_{N,0}^{(2q+1)/2p} = N\lambda_{N,0}^{(2p+2q+1)/2p}\lambda_{N,0}^{-1} \to \infty.$$

By Lemma 3.5, $T^1_{N,q}(\lambda_{N,0}) \to 0$ and

$$T^2_{N,q}(\lambda_{N,0}) \leq \frac{C}{N\lambda_{N,0}^{(2p+2q+1)/2p}}$$

so that $T^2_{N,q}(\lambda_{N,0}) \to 0$ by Lemma 3.6. ∎

Proposition 3.2 says that even though $\lambda_{N,0}$ and $\lambda_{N,q}$ asymptotically split apart (see Corollary 3.2), the regularized solution corresponding to $\lambda_{N,0}$ (which converges to the true one in the L_2-norm as proved in [6]) converges however, to the same solution in the H_q-norm and indeed minimizes the error asymptotically.

REMARK 3.1. The result of convergence could have been obtained directly by observing that the bounds (10) and (11) satisfied by $\lambda_{N,0}$ meet the requirements of Lemma 3.2 for convergence in H_p-norm.

4. GCV

Let $\lambda_{N,V}$ be the minimizer of the expected value of the GCV criterion (5).

LEMMA 4.1. *If $F \in H_{p+q}$, then*

$$N\lambda_{N,V}^{(2p+2q+1)/2p} \to \infty.$$

PROOF. We have

$$T_{N,0}(\lambda) = \sigma^2 (1 - 2B_N(\lambda)) + B^2_N(\lambda)V_N(\lambda) \tag{13}$$

with

$$B_N(\lambda) = \frac{1}{N}\sum_{j=-1}^{J-1} a_j \frac{a_j^{2p}}{1 + a_j^{2p}\lambda} = 1 - \frac{1}{N}\sum_{j=-1}^{J-1} \frac{a_j}{1 + a_j^{2p}\lambda}.$$

Hence,

$$B'_N(\lambda) = \frac{1}{N}\sum_{j=-1}^{J-1} \frac{a_j^{2p+1}}{\left(1 + a_j^{2p}\lambda\right)^2}.$$

By Lemma 3.1,

$$0 \leq 1 - B_N(\lambda) \leq \frac{C}{N\lambda^{1/2p}}, \qquad 0 \leq B'_N(\lambda) \leq \frac{C}{N\lambda^{(2p+1)/2p}}.$$

From equation (13),

$$T'_{N,0}(\lambda) = -2\sigma^2 B'_N(\lambda) + 2B_N(\lambda)B'_N(\lambda)V_N(\lambda) + B^2_N(\lambda)V'_N(\lambda).$$

Multiply both sides by $B_N(\lambda)$ and replace $B^2_N(\lambda)V_N(\lambda)$ by using equation (13). We obtain

$$B_N(\lambda)T'_{N,0}(\lambda) = 2B'_N(\lambda)T_{N,0}(\lambda) - 2\sigma^2 B'_N(\lambda)(1 - B_N(\lambda)) + B^3_N(\lambda)V'_N(\lambda).$$

Since $V_N'(\lambda_{N,V}) = 0$, we have

$$\begin{aligned} T^1_{N,0}(\lambda_{N,V}) &= T^2_{N,0}(\lambda_{N,V}) - \frac{2\sigma^2 B_N'(\lambda_{N,V})\,(1 - B_N(\lambda_{N,V}))}{B_N(\lambda_{N,V})} \\ &\quad + \frac{2B_N'(\lambda_{N,V}) T_{N,0}(\lambda_{N,V})}{B_N(\lambda_{N,V})} \\ &\geq T^2_{N,0}(\lambda_{N,V}) - \frac{2\sigma^2 B_N'(\lambda_{N,V})\,(1 - B_N(\lambda_{N,V}))}{B_N(\lambda_{N,V})}. \end{aligned}$$

We know from [6, Theorem 4.5] that $N\lambda_{N,V}^{1/2p} \to \infty$. Hence, one can prove as in Lemma 3.5 that

$$T^2_{N,0}(\lambda_{N,V}) \geq \frac{C}{N\lambda_{N,V}^{(2p+1)/2p}}.$$

As $(1 - B_N(\lambda_{N,V}))/B_N(\lambda_{N,V}) \to 0$ it follows that

$$T^2_{N,0}(\lambda_{N,V}) - \frac{2\sigma^2 B_N'(\lambda_{N,V})}{B_N(\lambda_{N,V})}(1 - B_N(\lambda_{N,V})) \geq \frac{C}{N\lambda_{N,V}^{(2p+1)/p}}$$

for N sufficiently large. We have thus proved that

$$T^1_{N,0}(\lambda_{N,V}) \geq \frac{C}{N\lambda_{N,V}^{(2p+1)/p}}.$$

The proof is concluded as in Lemma 3.6. ∎

PROPOSITION 4.1. *If $F \in H_{p+q} \setminus \{0\}$, then*

$$T_{N,q}(\lambda_{N,V}) \to 0 \quad \text{and} \quad T'_{N,q}(\lambda_{N,V}) \to 0.$$

PROOF. Same as for Proposition 3.2 using Lemma 4.1 instead of Lemma 3.6. ∎

Proposition 4.1 says that the regularized solution corresponding to the usual GCV criterion (which as shown in [6] minimizes the L_2-norm error) also minimizes the H_q-norm error asymptotically.

5. NUMERICAL EXPERIMENTS

In the present section, we work some numerical experiments in order to show performance of the regularization method with respect to the Sobolev norm. This is important from the practical point of view due to Corollary 3.2, where it is stated that $\lambda_{N,0}$ and $\lambda_{N,q}$ asymptotically split apart. We shall consider a test function (Doppler) available in the literature [2] and the Gaussian function $F(x) = \exp(-(x - 0.5)^2)$. The following error index, $I(\lambda)$, is considered in order to estimate the actual accuracy of the regularization method:

$$I(\lambda) = \sqrt{\sum_{j=0}^{J-1} \sum_{k=0}^{a_j-1} a_j^{2jq} \left(\frac{Y_{j,k}}{1 + a_j^{2p}\lambda} - F_{j,k} \right)^2} \simeq \left\| F^{N,\lambda} - F^N \right\|_{H^q}.$$

First, we notice from Corollary 3.2 that, even though $\lambda_{N,0}$ and $\lambda_{N,q}$ asymptotically split apart, this happens slowly however, as shown by (12) for the worst case (that is for $p \simeq q$) $\lambda_{N,q}/\lambda_{N,0} \approx N^{1/6}$. In Tables 1 and 2, we show for all test functions and several values of N the index $I(\lambda)$ corresponding to $\lambda_{N,0}$ and $\lambda_{N,q}$. In all computations the values $p = 2$, $q = 1.5$ were considered. For comparison, we also show the value of the index $I(\lambda)$ in the case no regularization is applied.

Table 1. Index $I(\lambda)$ evaluated al $\lambda = 0$, $\lambda_{N,q}$, $\lambda_{N,0}$, and $\lambda_{N,V}$ for the Gaussian test function and several values of N. For all cases $p = 2$ and $q = 1.5$.

n	$\lambda = 0$	$\lambda_{N,q}$	$\lambda_{N,0}$	$\lambda_{N,V}$
2^7	0.10E1	0.18	0.18	0.50
2^8	0.11E1	0.19	0.19	0.38
2^9	0.11E1	0.20	0.21	0.20
2^{10}	0.11E1	0.22	0.22	0.23
2^{12}	0.11E1	0.25	0.25	0.25
2^{14}	0.12E1	0.28	0.28	0.29
2^{16}	0.12E1	0.31	0.31	0.31
2^{18}	0.13E1	0.33	0.34	0.34
2^{20}	0.13E1	0.36	0.36	0.36

Table 2. Index $I(\lambda)$ evaluated al $\lambda = 0$, $\lambda_{N,q}$, $\lambda_{N,0}$, and $\lambda_{N,V}$ for the test function Doppler and several values of N. For all cases $p = 2$ and $q = 1.5$.

n	$\lambda = 0$	$\lambda_{N,q}$	$\lambda_{N,0}$	$\lambda_{N,V}$
2^7	0.18E3	0.17E3	0.17E3	0.18E3
2^8	0.15E4	0.11E4	0.11E4	0.15E4
2^9	0.15E5	0.62E4	0.64E4	0.15E5
2^{10}	0.12E6	0.18E5	0.18E5	0.12E6
2^{12}	0.77E7	0.31E5	0.31E5	0.77E7
2^{14}	0.50E9	0.30E5	0.30E5	0.50E9
2^{16}	0.33E11	0.26E5	0.26E5	0.33E11
2^{18}	0.21E13	0.21E5	0.21E5	0.21E13
2^{20}	0.13E15	0.16E5	0.16E5	0.13E15

The convergence of the method can be observed for the regularization method (see column $\lambda_{N,0}$), even though slower; moreover, the digits closely approach the best achievable value (compare column $\lambda_{N,0}$ with $\lambda_{N,q}$).

The effectiveness of the GCV criterion in choosing the regularization parameter is also shown in the same tables, where the index $I(\lambda)$ is also computed at $\lambda_{N,V}$. As for $\lambda_{N,0}$, the digits of column $\lambda_{N,V}$ also approach column $\lambda_{N,q}$.

6. CONCLUSIONS

The present paper dealt with the wavelet regularization method for smoothing data already developed by the authors. The point addressed here was the convergence of the derivatives of the function retrieved from a noised sample to the derivatives of the true function. This problem is very interesting in applications (e.g., image processing), since derivatives are useful for many purposes, as picking boundaries, estimating curvatures and so on. It has been proved that the error of the p^{th} order regularization method developed by the authors asymptotically vanishes in the average with respect to the H^q-norm, $0 \leq q < p$. Even though the corresponding regularization parameter split apart from the best achievable regularization parameter, however, both minimize asymptotically the H^q-error in the average. It has been also proved that the regularization parameter as estimated by the GCV criterion also asymptotically minimizes the H^q-error in the average. Then wavelet regularization endowed with the GCV criterion provides thus an effective way to smooth noisy data and to retrieve the true function and its derivatives. Numerical experiments were worked out in order to show performance of the method and that the (slow) splitting of the regularization parameter as estimated by the method from the optimal parameter does not affect performance appreciably from an applied point of view.

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